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The superconducting penetration depth from the semiclassical model

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Abstract. The semiclassical approach of Onsager and Pippard has been very successful in relating the electronic properties of a normal metal to its band structure and fermi surface. This paper extends the method to a superconductor. A generalized London equation relating the supercurrent density to the vector potential is obtained, in terms of band and fermi surface parameters and an energy gap which may be anisotropic. The results allow an interpretation of measured penetration depths directly in terms of the electronic and gap structures of the superconductor. They lead to easy physical visualization of the origin of anisotropies and other features of measured penetration depths.

Keywords: Real metals and superconductors; Generalized London equation; Penetration depth anisotropy.

I Introduction

The standard theory [1] of the temperature-dependent penetration depth $\lambda(T)$ in a superconductor assumes a spherical fermi surface with the dispersion relation

$$\xi_k = \frac{\hbar^2 k^2}{2m} - \mu \tag{1}$$

where ξ_k, k, μ, and m are respectively the energy (measured from the fermi surface), wave number, chemical potential and mass of the electron. In a real superconductor, the ionic potential modifies the dispersion relation $\varepsilon_k(\mathbf{k})$ and leads in general to a non-spherical fermi surface whose equation is

$$\xi_k(\mathbf{k}_F) \equiv \varepsilon_k(\mathbf{k}_F) - \mu = 0 \tag{2}$$

where $\mathbf{k}_F$ is the fermi wave vector. In this situation it is not immediately evident what should be taken for the mass m and the number density n_s which appear in the London formula for the superconducting penetration depth

$$\lambda_L = \left(\frac{mc^2}{4\pi n_s e^2}\right)^{1/2} . \tag{3}$$

A further problem arises with non-cubic superconductors, where the penetration depth varies with the direction of the shielding current with respect to the crystal axes, and shows crystalline anisotropy. This anisotropy is caused by a combination of the anisotropies of the fermi surface and the superconducting energy gap. A striking and widely studied example is provided by the cuprate superconductors. Experiments show that $\lambda_c/\lambda_{ab} \cong 5$ in the yttrium-based compounds for shielding current flowing respectively in the c-direction and ab-plane of the crystal. It has become the practice [2] to associate this anisotropy with anisotropic effective masses m_c and m_{ab} in the London formula.

This practice carries the implicit assumption that the dispersion relation for the electrons in such superconductors is ellipsoidal:

$$\xi_k = \frac{\hbar^2}{2}\left(\frac{k_{ab}^2}{m_{ab}} + \frac{k_c^2}{m_c}\right) - \mu \tag{4}$$

where k_{ab} and k_c are the components of $\mathbf{k}$ in the ab-plane and along the c-axis respectively. Band structure calculations [3], however, lead to a totally different picture: the fermi surface is in several sheets, none of which even approximates to an ellipsoidal shape. It is clear therefore that one should interpret the measured penetration depths, and their anisotropy and temperature dependence, in terms of the actual dispersion relation $\varepsilon_k(\mathbf{k})$ as given by band structure calculations, and given gap anisotropy $\Delta_k(\mathbf{k})$. This holds of course for all superconductors.

A similar problem arises in the treatment of the electronic properties of a normal metal with arbitrary ε_k and non-ellipsoidal fermi surfaces. A first-principles treatment of the response of such a system to applied electric and especially magnetic fields encounters formidable difficulties [4]. An alternative, semiclassical, approach was proposed by Onsager [5], and applied by Chambers, Pippard, and others [6] to an analysis of magnetotransport and Landau quantization effects in real metals. We extend this method here to evaluate the current response in a real superconductor; the results show clearly how, for example, the penetration depth λ is determined by the characteristics of the fermi surface and the energy gap in the superconductor. The method permits an easy visualization of the phenomena in $\mathbf{k}$-space and should appeal to experimenters concerned with such measurements. For those who are more theoretically inclined, we might mention that the electromagnetic response of a real metal in both the normal and superconducting states has been treated from a microscopic kinetic equation approach, leading to the same results as here [7].

We illustrate the semiclassical approach in Section II by using it, following Pippard [6], to calculate the electrical conductivity of a normal metal, which reduces to the familiar free-electron result for the corresponding ε_k. We extend the approach to a superconductor in Section III, and obtain a generalized London equation relating the supercurrent density to the vector potential. The equation includes a backflow term in order to conserve charge and to be compatible with gauge invariance. Some implications of this equation are discussed in Section IV, and conclusions and a summary are presented in Section V.

II The semiclassical approach: The normal conductivity

In order to illustrate how the semiclassical approach works, we apply it in this section to calculate the electrical conductivity tensor for a metal with an arbitrary fermi sur-

face. A thorough discussion of this can be found in reference [6]. We present a brief account here for the benefit of those of today's workers in superconductivity who may not be familiar with this method.

We assume that the anisotropic dispersion relation $\varepsilon_k(\mathbf{k})$, the anisotropic scattering time $\tau_k(\mathbf{k})$, and the chemical potential μ for the electrons are known. An electron of mean wave vector $\mathbf{k}$ is represented by a wave packet of Bloch waves centered at $\mathbf{k}$, and its velocity $\mathbf{v}_k$ is given by the velocity of the wave packet,

$$\mathbf{v}_k = \frac{1}{\hbar}\nabla_k \varepsilon_k \ . \tag{5}$$

The fundamental assumption of the method is that the application of electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ changes the $\mathbf{k}$-vector of each electron according to

$$\hbar\left(\frac{d}{dt}+\frac{1}{\tau_k}\right)\mathbf{k} = e\left(\mathbf{E}+\frac{1}{c}\mathbf{v}_k\times\mathbf{B}\right) \ . \tag{6}$$

The justification of this equation must be found in a solution of the full quantum-mechanical problem of the motion of Bloch electrons in the applied fields. Ashcroft and Mermin [8] discuss this matter, and it is interesting to quote from them: "*The reader who is dissatisfied with the very incomplete and merely suggestive bases we shall offer for the semiclassical model is urged to examine the broad array of mysteries and anomalies of free electron theory that the model resolves. Perhaps a suitable attitude to take is this: If there were no underlying microscopic quantum theory of electrons in solids, one could still imagine a semiclassical mechanics (guessed by some late nineteenth-century Newton of crystalline spaces) that was brilliantly confirmed by its account of observed electronic behaviour, just as classical mechanics was confirmed by its accounting for planetary motion, and only very much later given a more fundamental derivation as a limiting form of quantum mechanics.*"

We now proceed to calculate the current density $\mathbf{j}$ produced by an electric field $\mathbf{E}$. Equation (6) can be integrated to give

$$\delta\mathbf{k}_E = \frac{e\mathbf{E}}{\hbar}\tau_k \tag{7}$$

for the change in $\mathbf{k}$ due to the field. Consider a volume element d^3k of $\mathbf{k}$-space at wave vector $\mathbf{k}$. We assume unit volume of sample. Then the number of electrons in d^3k before the field is applied is dn,

$$dn = \frac{1}{4\pi^3}f(\xi_k)d^3k \tag{8}$$

where $f(\xi_k)$ is the fermi distribution function:

$$f(\xi_k) = (e^{\xi_k/k_BT}+1)^{-1} \tag{9}$$

and $\xi_k \equiv \varepsilon_k - \mu$. The contribution of these electrons to the current is given by $dn\cdot e\mathbf{v}_k$. Another volume element d^3k located at $-\mathbf{k}$ contains exactly the same number of elec-

trons, since the dispersion relation is symmetric in $\mathbf{k}$. These two volume elements make equal and opposite contributions to the current, because the electron velocities are antisymmetric in $\mathbf{k}$. Thus the total current, obtained by summing over $\mathbf{k}$, is zero in the absence of a field, as is of course to be expected.

The picture changes when the field $\mathbf{E}$ is applied. As a consequence of Eq. (4), each electron state is shifted by $\delta\mathbf{k}_E$. Therefore, the electrons which now find themselves in the element d^3k at wave vector $\mathbf{k}$ are the ones which were in an equal element at $\mathbf{k}-\delta\mathbf{k}_E$ before the field was applied, with energy $\varepsilon_k-\delta\varepsilon_k$, where

$$\delta\varepsilon_k = \left(\frac{\partial\varepsilon_k}{\partial\mathbf{k}}\right)\cdot\delta\mathbf{k}_E = \left(\frac{\partial\varepsilon_k}{\partial\mathbf{k}}\right)\cdot\left(\frac{e\mathbf{E}}{\hbar}\right)\tau_k = e\tau_k\mathbf{v}_k\cdot\mathbf{E}\ . \tag{10}$$

Similarly the element d^3k at $-\mathbf{k}$ is now occupied by those electrons which previously were in the element d^3k at $-(\mathbf{k}+\delta\mathbf{k}_E)$ with energy $\varepsilon_k+\delta\varepsilon_k$. Now the electron populations in the two volume elements are no longer equal, and their difference results in a net transport current $d\mathbf{j}_k$,

$$d\mathbf{j}_k = \frac{d^3k}{4\pi^3}(f(\varepsilon_k-\delta\varepsilon_k)-f(\varepsilon_k+\delta\varepsilon_k))e\mathbf{v}_k = \frac{d^3k}{2\pi^3}\left(-\frac{\partial f}{\partial\varepsilon_k}\right)\delta\varepsilon_k e\mathbf{v}_k$$

$$= \frac{d^3k}{2\pi^3}\left(-\frac{\partial f}{\partial\varepsilon_k}\right)e^2\tau_k(\mathbf{v}_k\mathbf{v}_k)\cdot\mathbf{E}\ . \tag{11}$$

The notation $(\mathbf{v}_k\mathbf{v}_k)$ represents the tensor product of two vectors. The total transport current is obtained by integrating over $\mathbf{k}$, introducing a factor $\frac{1}{2}$ to allow for the fact that each $\mathbf{k}$ contributes twice to the integral:

$$\mathbf{j} = \frac{e^2}{4\pi^3}\int d^3k\left(-\frac{\partial f}{\partial\varepsilon_k}\right)\tau_k(\mathbf{v}_k\mathbf{v}_k)\cdot\mathbf{E}\ . \tag{12}$$

In component form, we have

$$j_i = \sigma_{im}E_m$$

with

$$\sigma_{im} = \frac{e^2}{4\pi^3}\int d^3k\left(-\frac{\partial f}{\partial\varepsilon_k}\right)\tau_k v_{ki}v_{km}\ ;\quad i,m = x,y,z \tag{13}$$

where v_{ki} and v_{km} are the components of the velocity $\mathbf{v}_k$ of the electron of wave vector $\mathbf{k}$.

For a metal at temperatures $T \ll T_F$, the fermi temperature, we can use the approximation

$$-\frac{\partial f}{\partial\varepsilon_k} \equiv \delta(\varepsilon_k-\mu)$$

and convert the integral in Eq. (13) to an integral over the fermi surface:

$$\sigma_{im} = \frac{e^2}{4\pi^3\hbar} \oint dS_F \frac{v_{Fi} v_{Fm}}{v_F} \tau_F \ , \tag{14}$$

where dS_F is an element of fermi surface area, $\mathbf{v}_F$ the velocity, and τ_F the scattering time, all taken at fermi wave vector $\mathbf{k}_F$. Eq. (14) can be evaluated for given ε_k, τ_k, and μ. For a free-electron fermi surface with isotropic τ, Eq. (14) reduces to the familiar Drude result

$$\sigma = \frac{ne^2}{m}\tau$$

where n and m are the number density and mass of the electrons.

We now look in more detail at the general expression for the conductivity tensor, Eq. (14). We recall that the electron number density n is

$$n = \frac{1}{4\pi^3} \int d^3k f(\varepsilon_k)$$

and the reciprocal band mass tensor is

$$m_{im}^{-1} \equiv \alpha_{im} = \frac{\partial^2 \varepsilon_k}{\hbar^2 \partial k_i \partial k_m} \ .$$

Neither of these quantities appears even implicitly in the expression for the conductivity tensor. It is therefore not meaningful to talk, as has sometimes been done, about mass anisotropies in relation to anisotropic conductivities in general. The only exception is the case of ellipsoidal (including the special case of spherical) fermi surfaces, with isotropic relaxation time. Even in cubic crystals, where the conductivity is a scalar, the fermi surface is usually not spherical, and the concepts of electron effective mass and density are not very meaningful when considering the conductivity.

III The generalized London equation

We consider a London superconductor, with a local relation between the supercurrent density and the vector potential. We now need the analogue of Eq. (7) to describe the response of the electrons in the superconducting state to an applied vector potential **A**. We note that Eq. (7) is exact for free electrons, where the momentum **p** is equal to $\hbar$ **k**. It therefore seems reasonable to try an analogous generalization of the London equation which holds for a free electron superconductor:

$$m\mathbf{v}_s = -\frac{e}{c}\mathbf{A}$$

where $\mathbf{v}_s$ is the drift velocity of the superconducting electrons. Then the generalization to an arbitrary dispersion relation would be, by analogy with Eq. (7),

$$\delta\mathbf{k}_A = -\frac{e}{\hbar c}\mathbf{A} \ . \tag{15}$$

In words: The effect of a magnetic field on the Cooper-pair states in a superconductor is to change the **k**-vector of each state to $\mathbf{k}+\delta\mathbf{k}_A$, with $\delta\mathbf{k}_A$ given by Eq. (15).

The ultimate justification for Eq. (15) must flow out of a BCS theory for Bloch electrons in a magnetic field. This, to the best of our knowledge, has not yet been done. It seems reasonable to assume that the validity of Eq. (15) has the same basis as that of Eq. (7). Further, as will be shown below, the results derived from it, when applied to the special cases for which the BCS theory has been worked out, agree with those results. And it can be applied in a transparent way to real fermi surfaces, which are rarely free-electron-like.

We wish now to calculate the supercurrent density $\mathbf{j}_s$ at temperature T in a superconductor with electronic dispersion relation ε_k, due to an applied magnetic field derived from a vector potential **A**. We use a method which parallels the one used in Section II to calculate the normal conductivity. The supercurrent is carried only by the Cooper pairs, whose density depends on the temperature because of the existence of thermally excited quasiparticles.

Before proceeding with the calculation, we shall set down the assumptions on which the model is based:

(*i*) The single-particle energies ε_k in the superconducting state are the same as in the normal state.
(*ii*) Cooper pairs are formed between electrons in states $+\mathbf{k}$ and $-\mathbf{k}$. We shall ignore electron spin in what follows, since it is irrelevant for the transport supercurrent.
(*iii*) There is a temperature-dependent energy gap Δ_k, which may in general be anisotropic.
(*iv*) The energy E_k of a quasiparticle in state **k** is given by

$$E_k^2 = \xi_k^2 + \Delta_k^2 \ . \tag{16}$$

(*v*) The quasiparticle occupancy of the state **k** is $f(E_k)$,

$$f(E_k) = \frac{1}{e^{E_k/k_B T}+1} \ . \tag{17}$$

(*vi*) The occupancy of the single-particle state **k** in the superconducting state is n_k,

$$n_k = u_k^2 f(E_k) + v_k^2[1-f(E_k)] \tag{18}$$

where u_k^2 is the probability that the pair state at **k** is empty, and v_k^2 is the probability that it is occupied. Taking note of Eq. (17), we see that the two terms on the right of Eq. (18) are respectively the occupancies of the state **k** by quasiparticles and by pairs. Using the BCS results

$$u_k^2 = \frac{1}{2}\left(1+\frac{\xi_k}{E_k}\right) \quad \text{and} \quad v_k^2 = 1-u_k^2 \ , \tag{19}$$

we can rewrite Eq. (18) in the more transparent form

$$n_k = v_k^2 + \frac{\xi_k}{E_k} f(E_k) \ . \tag{20}$$

We note from Eq. (20) that, as one would expect, n_k changes smoothly from v_k^2 at $T=0$ to $f(\varepsilon_k)$ at $T=T_c$.

(*vii*) The application of a magnetic field $\mathbf{B}=\nabla\times\mathbf{A}$ changes the **k**-vector of each Cooper state to $\mathbf{k}+\delta\mathbf{k}_A$ with $\delta\mathbf{k}_A$ given by Eq. (15), and therefore the corresponding single-particle energy ε_k to $\varepsilon_k+\delta\varepsilon_k$, where

$$\delta\varepsilon_k=\frac{\partial\varepsilon_k}{\partial\mathbf{k}}\cdot\delta\mathbf{k}_A=-\frac{e}{c}\mathbf{v}_k\cdot\mathbf{A}\ . \tag{21}$$

We now proceed to calculate the supercurrent. The quasiparticles do not contribute to this current, and so the effective occupancy which produces the supercurrent is $(n_k-f(E_k))$, and this quantity multiplied by $(1/4\pi^3)d^3k\cdot e\mathbf{v}(\mathbf{k})$ is the contribution of the element d^3k to the supercurrent. Now consider two equal volume elements d^3k at $+\mathbf{k}$ and $-\mathbf{k}$ respectively. Before the magnetic field is applied, $(n_k-f(E_k))$ is the same in the two elements, with velocities $+\mathbf{v}_k$ and $-\mathbf{v}_k$ respectively, so that there is no net current.

When now the magnetic field is applied, the **k**-vector of each state is shifted to $\mathbf{k}+\delta\mathbf{k}_A$, resulting in an imbalance between the values of the quantity $(n_k-f(E_k))$ at $+\mathbf{k}$ and $-\mathbf{k}$ respectively, and therefore to a net current. The number dv_k of electrons in a volume d^3k at **k** contributing to the supercurrent is now

$$dv_k=\frac{d^3k}{4\pi^3}\left[n_k(\varepsilon_k-\delta\varepsilon_k)-f(E_k-\delta\varepsilon_k)\right]\ . \tag{22}$$

The equation uses the fact that the single-particle and excitation energies in the state at **k** in the current-carrying state are $(\varepsilon_k-\delta\varepsilon_k)$ and $(E_k-\delta\varepsilon_k)$ respectively.

The number of electrons in d^3k at $-\mathbf{k}$ which contribute to the supercurrent is similarly given by

$$dv_{-k}=\frac{d^3k}{4\pi^3}\left[n_k(\varepsilon_k+\delta\varepsilon_k)-f(E_k+\delta\varepsilon_k)\right]\ . \tag{23}$$

The total drift current, remembering that $\mathbf{v}_k=-\mathbf{v}_{-k}$, is obtained by adding the contributions from $+\mathbf{k}$ and $-\mathbf{k}$ and integrating over **k**:

$$\mathbf{j}_s=e\int(dv_k\mathbf{v}_k+dv_{-k}\mathbf{v}_{-k})=-\frac{e^2}{4\pi^3c}\int d^3k\left(-\frac{\partial n_k}{\partial\varepsilon_k}+\frac{\partial f(E_k)}{\partial E_k}\right)(\mathbf{v}_k\mathbf{v}_k)\cdot\mathbf{A}=-\mathbb{T}\cdot\mathbf{A} \tag{24}$$

where the symmetric tensor $\mathbb{T}$ is defined by

$$\mathbb{T}\equiv\frac{e^2}{4\pi^3c}\int d^3k\left(-\frac{\partial n_k}{\partial\varepsilon_k}+\frac{\partial f(E_k)}{\partial E_k}\right)(\mathbf{v}_k\mathbf{v}_k)\ . \tag{25}$$

A factor of $\frac{1}{2}$ has been included in Eq. (24) to allow for the double-counting of each **k**-state in the integral.

Eq. (24) is the generalization of the original London equation to a BCS superconductor with a general dispersion relation and at temperature T. But it is incomplete as it

stands, for it does not satisfy charge conservation. This is made clear by the following argument. Consider a superconductor occupying an infinite half-space bounded by a plane, and an applied magnetic field parallel to the plane. Then from symmetry the currents and fields can vary only in the direction perpendicular to the plane. Taking the wave vector $\mathbf{q}$ in this direction, we can write Eq. (24) in terms of Fourier components:

$$\mathbf{j}_q = -\mathbb{T}\cdot\mathbf{A}_q \ .$$

Charge conservation requires that

$$\nabla\cdot\mathbf{j} = 0 = i\mathbf{q}\cdot\mathbf{j}_q = -i\mathbf{q}\cdot\mathbb{T}\cdot\mathbf{A}_q \ . \tag{26}$$

But $\mathbf{q}\cdot\mathbb{T}\cdot\mathbf{A}_q$ is in general not equal to zero. We also note that since the magnetic field $\mathbf{B}$ is given by

$$\mathbf{B} = \nabla\times\mathbf{A} \ ,$$

a change of $\mathbf{A}$ to $\mathbf{A}+\nabla\chi$, where χ is a scalar function of position, leaves $\mathbf{B}$ unchanged. This *gauge transformation* in terms of Fourier components is

$$\mathbf{A}_q \rightarrow \mathbf{A}_q + i\chi_q\mathbf{q} \ .$$

This transformation however changes the current, which now becomes

$$\mathbf{j}_q = -\mathbb{T}\cdot(\mathbf{A}_q + i\chi_q\mathbf{q}) \ .$$

The function χ_q is fixed by requiring charge conservation:

$$i\mathbf{q}\cdot\mathbf{j}_q = 0 \ ,$$

which can be solved to give

$$i\chi_q = -\frac{\mathbf{q}\cdot\mathbb{T}\cdot\mathbf{A}}{\mathbf{q}\cdot\mathbb{T}\cdot\mathbf{q}}$$

and finally

$$\mathbf{j}_s = -\left(\mathbb{T} - \frac{(\mathbb{T}\cdot\mathbf{q})(\mathbf{q}\cdot\mathbb{T})}{\mathbf{q}\cdot\mathbb{T}\cdot\mathbf{q}}\right)\cdot\mathbf{A} \ . \tag{27}$$

Equation (27) is gauge-invariant and conserves charge. The second term on the right can be referred to as the *backflow*. We note that the backflow depends only on the direction of $\mathbf{q}$, and not on its magnitude.

Eqs. (25) and (27), together with the Maxwell equation

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{j}_s \tag{28}$$

complete the formal solution to the problem of a superconductor at temperature T, with anisotropic ε_k and Δ_k, in a steady magnetic field.

Since the tensor in Eq. (27) is symmetric, it can be diagonalized by a transformation to principal axes, which will coincide with axes of crystal symmetry for all cases of interest. We can then write Eq. (27) in component form as

$$j_i = -t_{ii} A_i \; ; \quad i = x,y,z \; . \tag{29}$$

We note that in general t_{xx}, t_{yy} and t_{zz} depend on the unit vector $\hat{\mathbf{q}}$.

IV Discussion

An examination of Eqs. (25) and (27) reveals the following features which can be easily proved:

(*a*) For a spherical fermi surface with electron mass m and number density n, and isotropic gap, Eq. (27) reduces at $T = 0$ to the nearly familiar form

$$\mathbf{j}_s = -\frac{ne^2}{mc}(\mathbf{1} - \hat{\mathbf{q}}\hat{\mathbf{q}}) \cdot \mathbf{A} \tag{30}$$

(where **1** is the unit matrix), from which the London formula, Eq. (3), follows in the Coulomb gauge

$$\nabla \cdot \mathbf{A} = 0 \; .$$

(*b*) For an ellipsoidal fermi surface with reciprocal mass tensor α_{ij} and electron density n, the result at $T = 0$ is, in component form,

$$j_i = -\frac{ne^2}{c}\left(\alpha_{ij} - \frac{\alpha_{ik} q_k q_l \alpha_{lj}}{q_m \alpha_{mn} q_n}\right) A_j$$

with summation over repeated indices, and this reduces, for the special geometry of (*d*) below and in the Coulomb gauge, to

$$j_i = -\frac{ne^2}{c} \alpha_{ii} A_i \; , \quad i = x,y,z \; . \tag{31}$$

(*c*) For a general fermi surface, the result cannot be expressed in terms of an electron band effective mass and density in any meaningful way.

(*d*) Referred to principal axes, and for a semi-infinite crystal bounded by a plane which contains the vector potential and two of these axes, say x and y, Eqs. (24) and (28) can be combined to give

$$\frac{\partial^2 A_x}{\partial z^2} = \frac{4\pi}{c} T_{xx} A_x \; , \quad \frac{\partial^2 A_y}{\partial z^2} = \frac{4\pi}{c} T_{yy} A_y \tag{32}$$

where we have used the fact that the backflow term vanishes in this geometry. Thus the components of $\mathbf{A}$, and consequently of $\mathbf{j}_s$ decay at different rates as one goes into the superconductor. This implies a gradual rotation of the two vectors $\mathbf{j}_s$ and $\mathbf{A}$ as z increases, in addition to their decay. One gets a simple exponential decay only if $\mathbf{A}$ is along one of the axes, and in this case one can define penetration depths

$$\lambda_{ii} = \left(\frac{c}{4\pi T_{ii}}\right)^{1/2}, \quad i = x, y, z \ . \tag{33}$$

It should be noted that λ_{ii} are not the components of either a vector or a tensor. We can *define* an effective masses m_{ii} through

$$m_{ii} \equiv \frac{ne^2}{cT_{ii}} \tag{34}$$

where n is the conduction electron density, and then we get the familiar London relation

$$\lambda_{ii} = \left(\frac{m_{ii}c^2}{4\pi ne^2}\right)^{1/2} \tag{35}$$

which has often been used to analyse experimental results. It should be noted that the m_{ii} so defined have nothing to do with band masses except in the cases (a) and (b) above; they are just a way of parametrizing experimental results which obscures their relation to the band structure of the metal. In fact, Eq. (33) shows that if one tries to interpret the penetration depth in terms of the London formula, one is forced to introduce an effective (m/n), with no discernible connection to the band mass or carrier concentration.

We now look at the structure of the tensor $\mathbb{T}$, Eq. (25). The integral consists of two terms which usually are called the diamagnetic and paramagnetic terms $\mathbb{T}_D$ and $\mathbb{T}_P$ respectively:

$$\mathbb{T} = \mathbb{T}_D - \mathbb{T}_P$$

with

$$\mathbb{T}_D = \frac{e^2}{4\pi^3 c} \int d^3k \left(-\frac{\partial n_k}{\partial \varepsilon_k}\right) (\mathbf{v}_k \mathbf{v}_k) \tag{36}$$

and

$$\mathbb{T}_P = \frac{e^2}{4\pi^3 c} \int d^3k \left(-\frac{\partial f(E_k)}{\partial E_k}\right) (\mathbf{v}_k \mathbf{v}_k) \ . \tag{37}$$

We note that in each integrand the derivative is appreciably different from zero only for values of $|\varepsilon_k - \mu| <$ a few times Δ_k, and we also have $\Delta_k \ll \mu$. We can therefore to an excellent approximation replace the tensor $\mathbf{v}_k \mathbf{v}_k$ with its value at $\varepsilon_k = \mu$, namely $\mathbf{v}_F \mathbf{v}_F$. We can for the same reason write

$$d^3k = \frac{dS_F d\varepsilon_k}{\hbar v_F}$$

where dS_F and v_F are respectively a constant-energy surface element and the magnitude of the electron velocity at $\varepsilon_k = \mu$. We then find

$$\mathbb{T}_D = \frac{e^2}{4\pi^3\hbar c} \oint dS_F \int_0^\infty d\varepsilon_k \left(-\frac{\partial n_k}{\partial \varepsilon_k}\right) \frac{\mathbf{v}_F \mathbf{v}_F}{v_F} \equiv \frac{e^2}{4\pi^3\hbar c} \oint dS_F \frac{\mathbf{v}_F \mathbf{v}_F}{v_F} \tag{38}$$

and

$$\mathbb{T}_P \equiv 2 \cdot \frac{e^2}{4\pi^3\hbar c} \oint dS_F \frac{\mathbf{v}_F \mathbf{v}_F}{v_F} \int_{\Delta_k}^\infty dE_k \left(-\frac{\partial f(E_k)}{\partial E_k}\right) \frac{E_k}{\sqrt{(E_k^2 - \Delta_k^2)}} \,. \tag{39}$$

The factor of 2 in the last equation comes from changing the variable of integration from ε_k to E_k.

An inspection of Eqs. (38) and (39) leads to the following conclusions:

(*a*) $\mathbb{T}_D$ is independent of temperature, and reflects the anisotropy of ε_k, or equivalently of the fermi surface.
(*b*) $\mathbb{T}_P$ vanishes as the temperature T tends to zero, and rises monotonically to reach the value $\mathbb{T}_D$ at $T = T_c$.
(*c*) If the energy gap is isotropic, then the anisotropy of $\mathbb{T}_p$ is independent of temperature, and is the same as the anisotropy of $\mathbb{T}_D$.
(*d*) If the gap is anisotropic, then the anisotropy of $\mathbb{T}_P$ is affected by the anisotropies of both ε_k and Δ_k, and is temperature-dependent.

These conclusions can be rephrased in an obvious manner in terms of the penetration depths λ_{ii} which are the quantities of experimental interest.

V Summary and Conclusions

Using the semi-classical model, we have derived an expression for the supercurrent $\mathbf{j}_s$ as a function of the vector potential $\mathbf{A}$ for a superconductor with arbitrary dispersion relation ε_k and energy gap Δ_k. For the special cases of spherical or ellipsoidal fermi surfaces, the original London results are recovered. In the general case, however, it is shown that it is not meaningful to express the results in terms of effective masses of density of the electrons. Rather, experiments should be interpreted in terms of known or assumed features of the electronic dispersion relations, fermi surface and energy gap.

The importance of including the backflow term in an anisotropic superconductor had already been pointed out earlier [9]. It is derived here in the spirit of the semiclassical model. It arises simply by requiring gauge invariance, or equivalently conservation of charge, in an anisotropic superconductor with a spatially varying magnetic field.

A comparison of Eqs. (14) and (38) leads to the interesting conclusion that in those oxide superconductors where it is found that

$$\left(\frac{\lambda_2}{\lambda_1}\right)^{1/2}_{T=0} \cong \frac{\sigma_1}{\sigma_2}$$

where the subscripts 1 and 2 stand for two crystal axes, the scattering time τ_k must be nearly isotropic; the anisotropy of the conductivity, as that of λ, must be due to the anisotropy of the fermi surface. And if the anisotropy of the measured penetration depth varies with temperature, then one can conclude that the energy gap must be anisotropic.

It is instructive to state in words the physical content of Eqs. (38) and (39). At $T = 0$, the supercurrent response is determined solely by the diamagnetic component $\mathbb{T}_D$, which apart from a multiplicative constant is the integral over the fermi surface of the tensor product of the fermi velocity and the vector surface element. This current is reduced at finite temperatures by the paramagnetic component, due to the thermally excited quasiparticles which result from pair-breaking.

As an example of the usefulness of this picture, we note from the structure of the integrals for $\mathbb{T}$ that a flat portion of the fermi surface, if such exists, makes no contribution to the supercurrent in a direction parallel to it, and would therefore tend to increase the penetration depth λ. One can also see that relatively low values of the fermi velocity over a significant part of the fermi surface would also increase λ. We suggest that a combination of these factors is responsible for the rather large values of λ which have been reported in the cuprate superconductors, and also in the so-called heavy electron [10] superconductors.

Finally, we state the central conclusion of this work: measurements of penetration depths in superconductors should be interpreted in terms of their band structures, fermi surfaces and velocities, and energy gaps, rather than being parametrized into effective masses and carrier concentrations. Such an approach provides a deeper insight into the phenomena and also their relationships to other electronic properties.

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[10] The heavy electron (or sometimes heavy fermion) compounds were so named because their electronic specific heats, proportional to the density of states $N(\varepsilon_F)$ at the fermi surface, are very large. As $N(\varepsilon_F)$ for a free electron metal is $m k_F/(\pi^2\hbar^2)$, and as k_F does not vary much from metal to metal, it was supposed that the electrons in these compounds have a very high effective mass. However, $N(\varepsilon_F)$ for an arbitrary fermi surface is equal to $(1/4\pi^3\hbar)\oint dS_F/v_F$. A large value for $N(\varepsilon_F)$ therefore means that the electrons are unusually slow, rather than that they are unusually heavy. See also T.M. Rice, J. Mag., Mag. Mtls. **63, 64** (1987) 689